

HEEGAARD FLOER CORRECTION TERMS OF (+1)-SURGERIES ALONG $(2, q)$ -CABLINGS

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ABSTRACT. The Heegaard Floer correction term (d -invariant) is an invariant of rational homology 3-spheres equipped with a Spin^c structure. In particular, the correction term of 1-surgeries along knots in S^3 is a $(2\mathbb{Z})$ -valued knot concordance invariant d_1 . In this paper, we estimate d_1 for the $(2, q)$ -cable of any knot K . This estimate does not depend on the knot type of K . If K belongs to a certain class which contains all negative knots, then equality holds. As a corollary, we show that the relationship between d_1 and the Heegaard Floer τ -invariant is very weak in general.

1. INTRODUCTION

Throughout this paper we work in the smooth category, all manifolds are compact, orientable and oriented. If X is a closed 4-manifold, then $\text{punc } X$ denotes X with an open 4-ball deleted.

1.1. Correction term and $(2, q)$ -cablings. In [8], Ozsváth and Szabó introduced a rational homology cobordism invariant d for rational homology 3-spheres equipped with a Spin^c structure from Heegaard Floer homology theory. Here rational homology cobordism is defined as follows.

Definition 1. For two rational homology 3-spheres Y_i with Spin^c structure \mathfrak{t}_i ($i = 1, 2$), we say that (Y_1, \mathfrak{t}_1) is *rational homology cobordant to* (Y_2, \mathfrak{t}_2) if there exists an oriented cobordism W from Y_1 to Y_2 with $H_j(W; \mathbb{Q}) = 0$ ($j = 1, 2$) which can be endowed with a Spin^c structure \mathfrak{s} whose restrictions to the Y_i are the \mathfrak{t}_i ($i = 1, 2$).

This relation is an equivalence relation on the set of pairs (Y, \mathfrak{t}) where Y is a rational homology 3-sphere and \mathfrak{t} is a Spin^c structure on Y . Moreover, the connected sum operation endows the quotient set θ^c of this relation with the structure of an abelian group.

The invariant d is called the *correction term*. In particular, d is a group homomorphism from θ^c to \mathbb{Q} . Note that if Y is an integer homology 3-sphere, then Y has a unique Spin^c structure. Hence in this case, we may denote the correction term simply by $d(Y)$ and it is known that the value of the invariant becomes an even integer.

Here we remark that for the integer homology 3-sphere $S_1^3(K)$ obtained by $(+1)$ -surgery along a knot K , $d(S_1^3(K))$ is not only a rational homology cobordism invariant of $S_1^3(K)$, but also a knot concordance invariant of K . In fact, Gordon [3] proved that if two knots K_1 and K_2 are concordant, then $S_1^3(K_1)$ and $S_1^3(K_2)$ are integer homology cobordant, and this implies that $d(S_1^3(K_1)) = d(S_1^3(K_2))$. In the rest of the paper we denote $d(S_1^3(K))$ simply by $d_1(K)$ and investigate d_1 as a knot

concordance invariant. Note that d_1 is a map from the knot concordance group to $2\mathbb{Z}$, but not a group homomorphism.

While explicit formulas for some knot classes have been given (for instance, alternating knots [9] and torus knots [2]), calculating d_1 is difficult in general. The calculation of d_1 is studied in [11]. In this paper, we investigate d_1 of the $(2, q)$ -cabling $K_{2,q}$ of an arbitrary knot K for an odd integer $q > 1$. In particular, we give the following estimate of $d_1(K_{2,q})$.

Theorem 1. *For any knot K in S^3 and $k \in \mathbb{N}$, we have*

$$d_1(K_{2,4k\pm 1}) \leq -2k.$$

Moreover, if K bounds a null-homologous disk in $\text{punc}(n\overline{\mathbb{CP}^2})$ for some $n \in \mathbb{N}$, then this inequality becomes equality.

We note that d_1 of the $(2, 4k \pm 1)$ -torus knot $T_{2,4k\pm 1}$ is equal to $-2k$ [2, 9], and hence Theorem 1 implies that $d_1(K_{2,q}) \leq d_1(O_{2,q})$, where O is the unknot. If q is an odd integer with $q \leq 1$, then from the Skein inequality [11, Theorem 1.4] we have $-2 \leq d_1(K_{2,q}) \leq 0$, and so $d_1(K_{2,q})$ is either -2 or 0 . In this paper, we focus on the case where $q > 1$.

Next, we consider knots which bound null-homologous disks in $\text{punc}(n\overline{\mathbb{CP}^2})$. We first assume that K is obtained from the unknot by a sequence of isotopies and crossing changes from positive to negative as in Figure 1. In this case, since such crossing changes can be realized by attaching 4-dimensional (-1) -framed 2-handles to S^3 (giving rise to the $\overline{\mathbb{CP}^2}$ factors) and handle slides (yielding the capping surface) as in Figure 2, we have the desired disk with boundary K . (The disk is null-homologous because it is obtained from the initial capping disk of the unknot by adding, as a boundary connected sum, two copies of the core of each 2-handle with opposite sign.) This implies that our knot class contains any negative knot. On the other hand, if the Heegaard Floer τ -invariant $\tau(K)$ of K is more than 0, then K cannot bound such a disk in $\text{punc}(n\overline{\mathbb{CP}^2})$. This follows immediately from [10, Theorem 1.1].

1.2. Comparison with the τ -invariant. The *Heegaard Floer τ -invariant* τ is a knot concordance invariant defined by Ozsváth-Szabó [10] and Rasmussen [12]. In comparing the computation of d_1 and τ , Peters poses the following question.

Question 1 (Peters [11]). *What is the relation between $d_1(K)$ and $\tau(K)$? Is it necessarily true that*

$$|d_1(K)| \leq 2|\tau(K)|?$$

Krcatovich [7] has already given a negative answer to this question. In fact, he showed that for any positive even integer a , there exists a knot K which satisfies $\tau(K) = 0$ and $|d_1(K)| = a$. In this paper, we give a stronger negative answer.

Theorem 2. *For any even integers a and b with $a > b \geq 0$, there exist infinitely many knot concordance classes $\{[K^n]\}_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$,*

$$|d_1(K^n)| = a \text{ and } 2|\tau(K^n)| = b.$$

This theorem follows from Theorem 1 and the following theorem by Hom. These two theorems give the following contrast between d_1 and τ : for certain knots, $\tau(K_{2,q})$ depends on the choice of K , while $d_1(K_{2,q})$ does not depend.

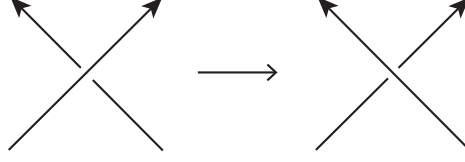


FIGURE 1.

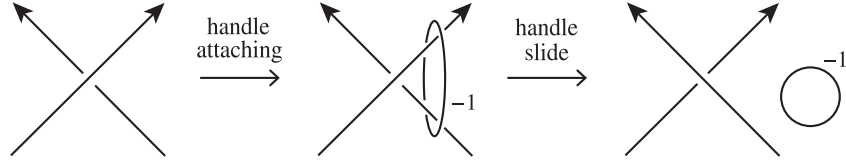


FIGURE 2.

Theorem 3 (Hom [5]). *Let K be a knot in S^3 and $p > 1$. Then $\tau(K_{p,q})$ is determined in the following manner.*

- (1) *If $\varepsilon(K) = 1$, then $\tau(K_{p,q}) = p\tau(K) + (p-1)(q-1)/2$.*
- (2) *If $\varepsilon(K) = -1$, then $\tau(K_{p,q}) = p\tau(K) + (p-1)(q+1)/2$.*
- (3) *If $\varepsilon(K) = 0$, then $\tau(K_{p,q}) = \tau(T_{p,q})$.*

Here $\varepsilon(K) \in \{0, \pm 1\}$ is a knot concordance invariant of K defined in [5].

1.3. An idea of proofs and another application. In this subsection, we describe how we obtain an estimate of $d_1(K_{2,q})$. We recall that if an integer homology 3-sphere Y bounds a negative definite 4-manifold, then $d(Y)$ satisfies the following inequality.

Theorem 4 (Ozsváth-Szabó [8]). *Let Y be an integer homology 3-sphere, then for each negative definite 4-manifold X with boundary Y , we have the inequality*

$$Q_X(\xi, \xi) + \beta_2(X) \leq 4d(Y)$$

for each characteristic vector ξ .

Here a 4-manifold X is called *negative (resp. positive) definite* if the intersection form Q_X of X is negative (resp. positive) definite. Moreover, $\xi \in H_2(X; \mathbb{Z})$ is called a *characteristic vector* if ξ satisfies $Q_X(\xi, v) \equiv Q_X(v, v) \pmod{2}$ for any $v \in H_2(X; \mathbb{Z})$, and β_i denotes the i -th Betti number.

Let us also recall the definition of d . For a rational homology 3-sphere Y with Spin^c structure \mathfrak{t} , the Heegaard Floer homologies $HF^*(Y, \mathfrak{t})$ ($*$ = +, -, ∞) are defined as an absolute \mathbb{Q} -graded $\mathbb{Z}[U]$ -modules, where the action of U decreases the grading by 2. These homology groups are related to one another by an exact sequence:

$$\cdots \rightarrow HF_{\bullet}^-(Y, \mathfrak{t}) \rightarrow HF_{\bullet}^{\infty}(Y, \mathfrak{t}) \xrightarrow{\pi} HF_{\bullet}^+(Y, \mathfrak{t}) \rightarrow HF_{\bullet-1}^-(Y, \mathfrak{t}) \rightarrow \cdots$$

Then $\pi(HF^{\infty}(Y, \mathfrak{t})) \subset HF^+(Y, \mathfrak{t})$ is isomorphic to $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$, and so we can define $d(Y, \mathfrak{t})$ to be the minimal grading of $\pi(HF^{\infty}(Y, \mathfrak{t}))$.

In order to prove our main results, we only use Theorem 4 without investigation of any Heegaard Floer homology groups. Indeed, the following theorem plays an essential role in this paper.

Theorem 5. *For any knot K and positive integer k , there exists a 4-manifold W which satisfies*

- (1) W is a simply-connected Spin manifold,
- (2) $\partial W = S_1^3(K_{2,4k\pm 1})$, and
- (3) $\beta_2(W) = \beta_2^+(W) = 8k$. In particular, W is positive definite.

Here $\beta_2^+(W)$ (resp. $\beta_2^-(W)$) denotes the number of positive (resp. negative) eigenvalues of Q_W . Theorem 5 implies that for any knot K , $-S_1^3(K_{2,4k\pm 1})$ bounds a negative definite Spin 4-manifold $-W$, and we obtain the inequality of Theorem 1 by applying Theorem 4 to the pair $(-W, -S_1^3(K_{2,4k\pm 1}))$.

We also mention that Tange [14] investigated which Brieskorn homology spheres bound a definite Spin 4-manifold, and also asked which integer homology 3-spheres bound a definite Spin 4-manifold. Theorem 5 gives a new construction of such 3-manifolds. Furthermore, for any knot K which bounds a null-homologous disk in $\text{punc}(n\overline{\mathbb{C}P^2})$ for some n , Theorems 1, 4 and 5 let us determine the value of $\mathfrak{ds}(S_1^3(K_{2,q}))$, where \mathfrak{ds} is an h -cobordism invariant of integer homology 3-spheres defined by Tange [14] as follows:

$$\mathfrak{ds}(Y) = \max \left\{ \frac{\beta_2(X)}{8} \left| \begin{array}{l} \partial X = Y \\ H_0(X) = \mathbb{Z}, H_1(X) = 0 \\ \beta_2(X) = |\sigma(X)|, w_2(X) = 0 \end{array} \right. \right\}.$$

Corollary 1. *If K bounds a null-homologous disk in $\text{punc}(n\overline{\mathbb{C}P^2})$, then*

$$\mathfrak{ds}(S_1^3(K_{2,4k\pm 1})) = k.$$

In addition, we also give the following result.

Proposition 1. *For any positive integer k , there exist infinitely many irreducible integer homology 3-spheres whose \mathfrak{ds} value is k .*

1.4. Further questions. It is natural to ask that if K does not bound a null-homologous disk in $\text{punc}(n\overline{\mathbb{C}P^2})$ for any n , then what are the possible $d_1(K_{2,q})$ values. As a case study on this question, we compute d_1 of the $(2, 2pq \pm 1)$ -cabling of the (p, q) -torus knot for $p, q > 0$, which does not bound a null-homologous disk in $\text{punc}(n\overline{\mathbb{C}P^2})$ for any n .

Proposition 2. *For any positive coprime integers p and q , we have*

$$d_1((T_{p,q})_{2,2pq\pm 1}) = d_1(T_{2,2pq\pm 1}).$$

This result is derived from [8, Proposition 8.1] and the fact that $(4pq \pm 1)$ -surgery along $(T_{p,q})_{2,2pq\pm 1}$ gives a lens space [1]. Based on this proposition and Theorem 1, we suggest the following question.

Question 2. *Does the equality $d_1(K_{2,q}) = d_1(T_{2,q})$ hold for all knots K ?*

Finally we suggest a question related to Proposition 1. While \mathfrak{ds} is an h -cobordism invariant, we will only prove that those infinitely many integer homology 3-spheres in Proposition 1 are not diffeomorphic to one another. Whether they are h -cobordant or not remains open. More generally, we suggest the following question.

Question 3. *Is it true that for any positive integer q and any two knots K and K' , $S_1^3(K_{2,q})$ and $S_1^3(K'_{2,q})$ are h -cobordant?*

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2. PROOF OF THEOREM 5

In this section, we prove Theorem 5. To prove the theorem, we first prove the following lemma. Here we identify $H_2(\text{punc } X, \partial(\text{punc } X); \mathbb{Z})$ with $H_2(X; \mathbb{Z})$.

Lemma 1. *Let $X \cong S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $K \subset \partial(\text{punc } X)$ a knot. If K bounds a disk $D \subset \text{punc } X$ with self-intersection $-n < 0$ which represents a characteristic vector in $H_2(X; \mathbb{Z})$, then there exists a 4-manifold W which satisfies*

- (1) W is a simply-connected Spin 4-manifold,
- (2) $\partial W = S_1^3(K)$, and
- (3) $\beta_2(W) = \beta_2^+(W) = n$. In particular, W is negative definite

Proof. By assumption, K bounds a disk D in $\text{punc } X$ which represents a characteristic vector in $H_2(X; \mathbb{Z})$. Since $[D, \partial D]$ is a characteristic vector and $\sigma(X) = 0$, we have $-n = [D, \partial D] \cdot [D, \partial D] \in 8\mathbb{Z}$ (see [6, Section 3]).

By attaching $(+1)$ -framed 2-handle h^2 along K , and gluing D with the core of h^2 , we obtain an embedded 2-sphere S in $W_1 := X \cup h^2$ such that S represents a characteristic vector in $H_2(W_1; \mathbb{Z})$ and satisfies $[S] \cdot [S] = -n + 1 < 0$. We next take the connected sum $(W_2, S') = (W_1, S) \# (\#_{n-2}(\mathbb{C}P^2, \mathbb{C}P^1))$, and then (W_2, S') satisfies

- (1) $\partial W_2 = S_1^3(K)$,
- (2) $\beta_2^+(W_2) = n\beta_2^-(W_2) = 1$, and
- (3) $[S']$ is a characteristic vector and $[S'] \cdot [S'] = -1$.

Therefore there exists a Spin 4-manifold W which satisfies $W_2 = W \# \overline{\mathbb{C}P^2}$. We can easily verify that this 4-manifold W satisfies the assertion of Lemma 1. \square

We next prove the following lemma.

Lemma 2. *For any knot $K \subset \partial(\text{punc}(S^2 \times S^2))$ and $k \in \mathbb{N}$, $K_{2,4k\pm 1}$ bounds a disk D in $\text{punc}(S^2 \times S^2)$ with self-intersection $-8k$ which represents a characteristic vector.*

Proof. For a given $K_{2,4k\pm 1}$, we apply the hyperbolic transformation shown in Figure 3 and Figure 4 to $2k$ full twists and a ± 1 half twist of $K_{2,4k\pm 1}$ respectively. Then we have a concordance A (with genus zero) in $S^3 \times [0, 1]$ from $K_{2,4k\pm 1} \subset S^3 \times \{0\}$ to the link $L \subset S^3 \times \{1\}$ shown in Figure 5. Note that the 4-manifold X shown in Figure 6 is diffeomorphic to $S^2 \times S^2$, and L is the boundary of $2k + 2$ disks E in X with the 0-handle deleted, where E consists of 2 copies of the core of h_1^2 and $2k$ copies of the core of h_2^2 . By gluing $(S^3 \times [0, 1], A)$ and $(\text{punc } X, E)$ along (S^3, L) , we obtain a disk D in $\text{punc } X$ with boundary $K_{2,4k\pm 1}$. It is easy to see that D represents a characteristic vector and has the self-intersection $-8k$. \square

Proof of Theorem 5. By Lemma 2, $K_{2,4k\pm 1}$ bounds a disk D in $\text{punc}(S^2 \times S^2)$ with self-intersection $-8k$ which represents a characteristic vector. Then, by applying Lemma 1 to the pair $(\text{punc}(S^2 \times S^2), D)$, we obtain the desired 4-manifold. \square

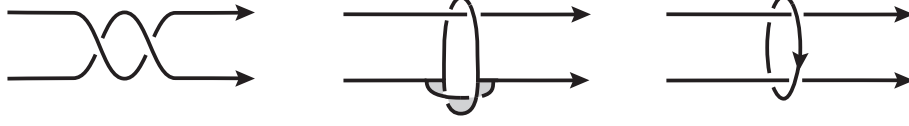


FIGURE 3.



FIGURE 4.

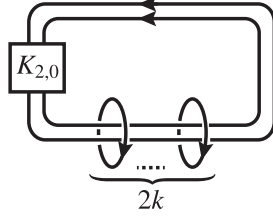


FIGURE 5.

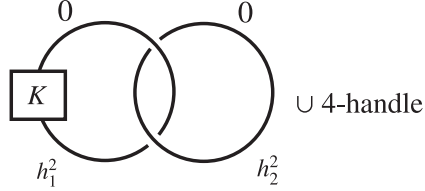


FIGURE 6.

3. PROOF OF THEOREM 1, THEOREM 2, AND COROLLARIES

In this section, we prove Theorem 1, Theorem 2, and two corollaries. We first prove Theorem 1, and then Corollary 1 immediately follows from Theorem 1.

Proof of Theorem 1. Let W be a 4-manifold with boundary $S_1^3(K_{2,4k\pm 1})$ which satisfies the assertion of Theorem 5. Since W is a Spin 4-manifold, the trivial element of $H_2(W; \mathbb{Z})$ is a characteristic vector. By Theorem 4, we have

$$0 + 8k \leq 4d(-S_1^3(K_{4k\pm 1})).$$

Since $d(-S_1^3(K_{4k\pm 1})) = -d(S_1^3(K_{4k\pm 1})) = -d_1(K_{4k\pm 1})$, this gives the inequality in Theorem 1.

Next, we suppose that K bounds a null-homologous disk D in $\text{punc } \overline{\mathbb{CP}^2}$ for some $n \in \mathbb{N}$. Excising a neighborhood of an interior point of D in $\text{punc } \overline{\mathbb{CP}^2}$, we obtain a null-homologous annulus A properly embedded in $\overline{\mathbb{CP}^2}$ with the 0-handle h^0 and the 4-handle h^4 deleted such that $(\partial h^0, A \cap \partial h^0)$ is the unknot and $(\partial h^4, A \cap \partial h^4)$ is K . Furthermore, since A is null-homologous, A gives a null-homologous annulus A' in $\overline{\mathbb{CP}^2} \setminus (h^0 \cup h^4)$ such that $(\partial h^0, A' \cap \partial h^0)$ is $T_{-2,4k\pm 1}$ and $(\partial h^4, A' \cap \partial h^4)$ is

$K_{2,4k\pm 1}$. Now we attach a $(+1)$ -framed 2-handle h^2 along $K_{2,4k\pm 1}$, and remove a neighborhood of a disk D' from $\overline{\mathbb{CP}^2} \setminus (h^0 \cup h^4)$, where D' is a disk obtained by gluing A' with the core of h^2 . Then we have a negative definite 4-manifold W with boundary $S_1^3(K_{2,4k\pm 1}) \amalg -S_1^3(T_{2,4k\pm 1})$. To see this, we regard a neighborhood of D' as a $(+1)$ -framed 2-handle along the mirror image of $T_{-2,4k\pm 1}$, i.e., $T_{2,4k\pm 1}$. If we denote the union of h^0 and this 2-handle by X , then $\overline{W} := (\overline{\mathbb{CP}^2} \setminus h^4) \cup h^2$ can be regarded as the 4-manifold obtained by gluing X to W along $-S_1^3(T_{2,4k\pm 1})$ (see Figure 7). In addition, the boundary of \overline{W} is $S_1^3(K_{2,4k\pm 1})$, hence the boundary of W is the disjoint union of $S_1^3(K_{2,4k\pm 1})$ and $-S_1^3(T_{2,4k\pm 1})$. The negative definiteness of W follows from the fact that the inclusion maps induces the isomorphism among the second homologies $(i_X)_* + (i_W)_* : H_2(X; \mathbb{Z}) \oplus H_2(W; \mathbb{Z}) \cong H_2(\overline{W}; \mathbb{Z})$ and the intersection forms $Q_X \oplus Q_W = Q_{\overline{W}}$, and $\beta_2^+(\overline{W}) = \beta_2^+(X) = 1$.

We apply Theorem 4 to the pair $(W, S_1^3(K_{2,4k\pm 1}) \amalg -S_1^3(T_{2,4k\pm 1}))$. Let $\gamma \in H_2(\overline{W}; \mathbb{Z})$ be the generator of h^2 and $\overline{\gamma}_1, \dots, \overline{\gamma}_n \in H_2(\overline{W}; \mathbb{Z})$ the generators induced from $H_2(n\overline{\mathbb{CP}^2} \setminus h^4; \mathbb{Z})$ which satisfy $Q_{\overline{W}}(\overline{\gamma}_i, \overline{\gamma}_j) = -\delta_{ij}$ (Kronecker's delta). Then the tuple $\gamma, \overline{\gamma}_1, \dots, \overline{\gamma}_n$ is a basis of $H_2(\overline{W}; \mathbb{Z})$ and gives a representation matrix

$$Q_{\overline{W}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Moreover, we see $\text{Im}(i_X)_* = \mathbb{Z}\langle \gamma \rangle$, and this gives $\text{Im}(i_W)_* = \mathbb{Z}\langle \overline{\gamma}_1, \dots, \overline{\gamma}_n \rangle$. Hence we can identify $(H_2(W; \mathbb{Z}), Q_W)$ with the pair of $\mathbb{Z}\langle \overline{\gamma}_1, \dots, \overline{\gamma}_n \rangle$ and the intersection form

$$\begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}.$$

Now we apply Theorem 4 to the tuple $(W, S_1^3(K_{2,4k\pm 1}) \amalg -S_1^3(T_{2,4k\pm 1}), \sum_{i=1}^n \overline{\gamma}_i)$, and we have the following inequality

$$(1) \quad \sum_{i=1}^n (-1) + n \leq 4d(S_1^3(K_{2,4k\pm 1})) - 4d(S_1^3(T_{2,4k\pm 1})).$$

Since $T_{2,4k\pm 1}$ is an alternating knot, [9, Corollary 1.5] gives the equality $d(S_1^3(T_{2,4k\pm 1})) = -2k$. This equality reduces (1) to the inequality

$$-2k \leq d(S_1^3(K_{2,4k\pm 1})).$$

This completes the proof. \square

Proof of Corollary 1. Let K be a knot which bounds a null-homologous disk in $\text{punc}(n\overline{\mathbb{CP}^2})$. By Theorem 5, $S_1^3(K_{2,4k\pm 1})$ bounds a positive definite Spin 4-manifold which satisfies the conditions of \mathfrak{ds} . Hence we have $\mathfrak{ds}(S_1^3(K_{2,4k\pm 1})) \geq k$. Moreover, Theorem 1 and [14, Theorem 2.1(9)] give the inequality

$$\mathfrak{ds}(S_1^3(K_{2,4k\pm 1})) \leq |d(S_1^3(K_{2,4k\pm 1}))|/2 = k.$$

This completes the proof. \square

We next prove Proposition 1.

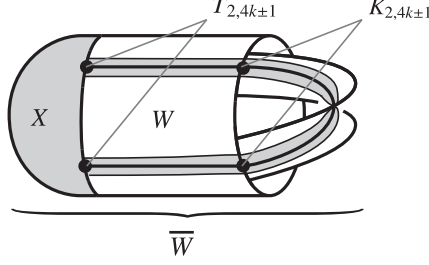


FIGURE 7.

Proof of Proposition 1. For any $n \in \mathbb{Z}$, let K_n be the knot shown in Figure 8. Since the cobordism in Figure 9 gives a null-homologous disk in $\text{punc } \mathbb{CP}^2$ with boundary K_n , it follows from Corollary 1 that the equality $\mathfrak{d}\mathfrak{s}(S_1^3((K_n)_{2,4k+1})) = k$ holds. We denote $S_1^3((K_n)_{2,4k+1})$ by $M_{n,k}$ and we will prove that $M_{n,k}$ is irreducible and if $m \neq n$, then $M_{m,k}$ is not diffeomorphic to $M_{n,k}$.

We recall that the Casson invariant of $S_1^3(K)$, denoted by $\lambda(S_1^3(K))$, is obtained from the following formula

$$\lambda(S_1^3(K)) = \frac{1}{2} \Delta_K''(1),$$

where $\Delta_K(t)$ is the normalized Alexander polynomial of K such that $\Delta_K(1) = 1$ and $\Delta_K(t) = \Delta_K(t^{-1})$ (see [13, Theorem 3.1]). It is easy to compute that for any knot K , $\Delta_{K_{2,4k+1}}''(1) = (\Delta_K(t^2) \cdot \Delta_{T_{2,4k+1}}(t))''|_{t=1} = 4\Delta_K''(1) + \Delta_{T_{2,4k+1}}''(1)$. Furthermore, we can easily compute that $\Delta_{K_n}(t) = nt - (2n - 1) + nt^{-1}$ and $\Delta_{K_n}''(1) = 2n$. These imply that if $m \neq n$, then

$$\lambda(M_{n,k}) - \lambda(M_{m,k}) = 2\Delta_{K_n}''(1) - 2\Delta_{K_m}''(1) = 4n - 4m \neq 0,$$

and hence $M_{m,k}$ is not diffeomorphic to $M_{n,k}$.

We next prove that $M_{n,k}$ is irreducible. The transformation shown in Figure 10 implies that for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, $(K_n)_{2,4k+1}$ has the tunnel number at most 2, and hence $M_{n,k}$ has the Heegaard genus at most 3. By the additivity of the Heegaard genus, if $M_{n,k}$ can be decomposed to N_1 and N_2 , then either N_1 or N_2 has the Heegaard genus 1. Assume that N_1 has the Heegaard genus 1. Then N_1 is diffeomorphic to one of S^3 , $S^1 \times S^2$, and lens spaces. However, $M_{n,k}$ is an integer homology 3-sphere, and hence N_1 must be diffeomorphic to S^3 . This completes the proof. \square

Finally we prove Theorem 2.

Proof of Theorem 2. Suppose that K is a negative knot. Then by Theorem 1, we have $d_1(K_{2,4k\pm 1}) = -2k$ for any $k \in \mathbb{N}$. Furthermore, it follows from [4, Theorem 1.1] and [5] that $\tau(K) = -g(K)$ and $\varepsilon = -1$, where $g(K)$ denotes the genus of K . Hence by Theorem 3, we have

$$\tau(K_{2,4k+1}) = 2\tau(K) + 2k + 1$$

and

$$\tau(K_{2,4k-1}) = 2\tau(K) + 2k.$$

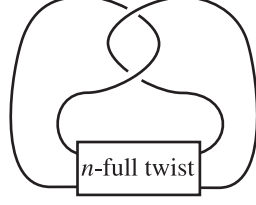


FIGURE 8.

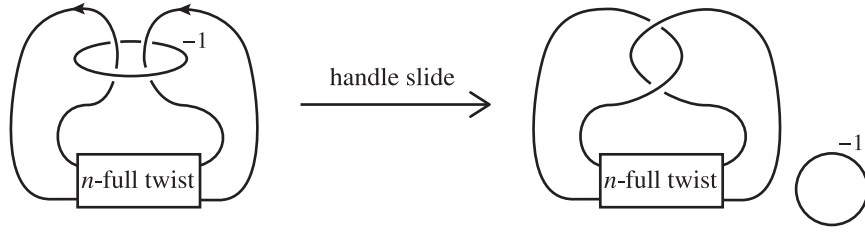


FIGURE 9.

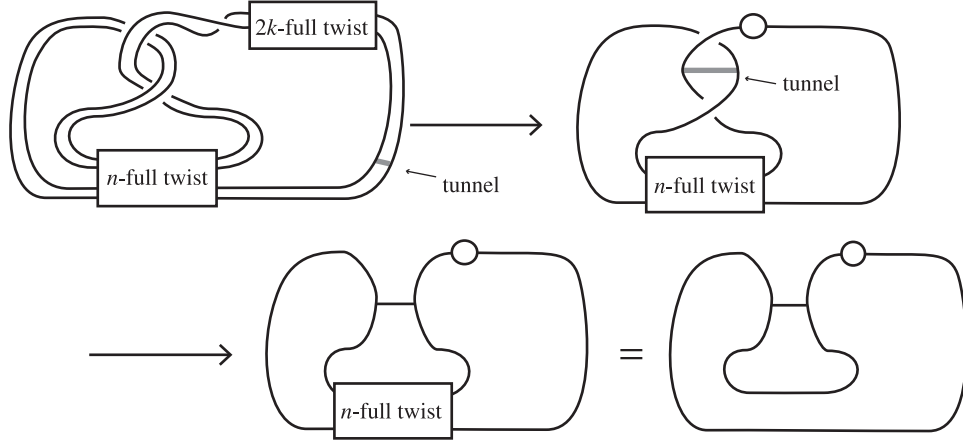


FIGURE 10.

For any $m, n \in \mathbb{N}$ and the knots $\{K_i\}_{i \in \mathbb{Z}}$ in Figure 8, we define

$$K^{m,n} := K_n \# K_{n+1} \# \cdots \# K_{n+m-1},$$

where $\#$ denotes the connected sum. Note that $K^{m,n}$ is a negative knot and has $g(K^{m,n}) = m$ for any $m, n \in \mathbb{N}$. Hence for any two even integers a and b with $a > b \geq 0$, if $b/2$ is odd, then we take $\{(K^{(2a-b+2)/4,n})_{2,2a+1}\}_{n \in \mathbb{N}}$ and we have

$$|d_1((K^{(2a-b+2)/4,n})_{2,2a+1})| = a$$

and

$$2|\tau((K^{(2a-b+2)/4,n})_{2,2a+1})| = 2|(-a + \frac{b}{2} - 1) + a + 1| = b.$$

Otherwise, we take $\{(K^{(2a-b)/4,n})_{2,2a-1}\}_{n \in \mathbb{N}}$ and we have

$$|d_1((K^{(2a-b)/4,n})_{2,2a-1})| = a$$

and

$$2|\tau((K^{(2a-b)/4,n})_{2,2a-1})| = 2|(-a + \frac{b}{2}) + a| = b.$$

It is easy to verify that if $l \neq n$, then $(K^{m,l})_{2,q}$ is not concordant to $(K^{m,n})_{2,q}$ for any $m \in \mathbb{N}$ and any odd integer $q > 1$. Actually, the Alexander polynomial $\Delta_{(K^{m,l})_{2,q} \# (K^{m,n})_{2,q}}(t) = \Delta_{K^{m,l}}(t^2) \cdot \Delta_{K^{m,n}}(t^2) \cdot (\Delta_{T_{2,q}}(t))^2$ is not of the form $f(t)f(t^{-1})$. This completes the proof. \square

4. PROOF OF PROPOSITION 2

In this last section, we prove Proposition 2. In order to prove it, we observe the normalized Alexander polynomial of cable knots, while we gave geometrical observations in the other sections.

We say that a Laurent polynomial $f(t)$ is *symmetric* if $f(t)$ satisfies $f(t) = f(t^{-1})$. Any symmetric Laurent polynomial $f(t)$ is the form of

$$f(t) = a_0(f) + \sum_{i=1}^d a_i(f)(t^i + t^{-i}).$$

We denote d by $\deg f$. Furthermore, the set of symmetric Laurent polynomials, denoted by S , is a \mathbb{Z} -submodule of $\mathbb{Z}[t, t^{-1}]$. we define $t_0(f)$ as

$$t_0(f) := \sum_{i=1}^{\deg f} i a_i(f).$$

Since a_i is a homomorphism from S to \mathbb{Z} for any i , t_0 is also a homomorphism. The following proposition is derived from [8, Proposition 8.1].

Proposition 3. *Let K be a knot in S^3 such that $S_p^3(K)$ is a lens space for some $p \in \mathbb{N}$, and $\Delta_K(t)$ the normalized Alexander polynomial of K . Then*

$$d_1(K) = -2t_0(\Delta_K).$$

It is shown in [1, Theorem 1] that $S_{4pq \pm 1}^3((T_{p,q})_{2,2pq \pm 1})$ is a lens space, and hence we only need to compute $t_0((T_{p,q})_{2,2pq \pm 1})$ to prove Proposition 2. In order to compute the value, we first prove the following lemma. Here we denote $t^i + t^{-i}$ by T_i .

Lemma 3. *For any symmetric Laurent polynomial $f(t)$ with $f(1) = 1$ and any integer $k \geq 1$, we have*

$$t_0(f(t) \cdot T_k) = \begin{cases} k & (k \geq \deg f) \\ ka_0(f) + \sum_{i=1}^k 2ka_i(f) + \sum_{i=k+1}^{\deg f} 2ia_i(f) & (1 \leq k < \deg f) \end{cases}.$$

Proof. Note that $T_i \cdot T_k = (t^i + t^{-i})(t^k + t^{-k}) = T_{i+k} + T_{i-k}$. This equality gives

$$f(t) \cdot T_k = a_0(f)T_k + \sum_{i=1}^{\deg f} a_i(f)T_{i+k} + \sum_{i=1}^{\deg f} a_i(f)T_{i-k}$$

and

$$t_0(f(t) \cdot T_k) = ka_0(f) + \sum_{i=1}^{\deg f} (i + k + |i - k|)a_i(f).$$

If $k \geq \deg f$, then we have

$$\begin{aligned} t_0(f(t) \cdot T_k) &= ka_0(f) + \sum_{i=1}^{\deg f} (i + k + k - i)a_i(f) \\ &= k(a_0(f) + \sum_{i=1}^{\deg f} 2a_i(f)) \\ &= kf(1). \end{aligned}$$

Since $f(1) = 1$, this gives the desired equality. It is easy to check that the equality for the case where $1 \leq k < \deg f$. \square

Proof of Proposition 2. We note that for any positive odd integer r ,

$$\begin{aligned} \Delta_{(T_{p,q})2,2r+1}(t) &= \Delta_{(T_{p,q})}(t^2) \cdot \Delta_{T_{2,2r+1}}(t) \\ &= (-1)^r \Delta_{(T_{p,q})}(t^2) \cdot (1 + \sum_{k=1}^r (-1)^k T_k). \end{aligned}$$

Hence if we set $t'_0 := (-1)^r t_0(\Delta_{(T_{p,q})2,2r+1}(t))$, then we have

$$(2) \quad t'_0 = t_0(\Delta_{T_{p,q}}(t^2)) + \sum_{k=1}^r (-1)^k t_0(\Delta_{T_{p,q}}(t^2) \cdot T_k).$$

We suppose that $r > \deg \Delta_{T_{p,q}}(t^2) =: d'$ and we set $a'_i := a_i(\Delta_{(T_{p,q})}(t^2))$ for $0 \leq i \leq d'$. Then it follows from Lemma 3 that

$$t_0(\Delta_{(T_{p,q})}(t^2)) = \sum_{i=1}^{d'} ia'_i$$

and

$$t_0(\Delta_{(T_{p,q})}(t^2) \cdot T_k) = \begin{cases} k & (k \geq d') \\ ka'_0 + \sum_{i=1}^k 2ka'_i + \sum_{i=k+1}^{d'} 2ia'_i & (1 \leq k < d') \end{cases}.$$

These equalities reduce (2) to

$$t'_0 = \begin{cases} (r/2)a'_0 + \sum_{i=1}^{d'/2} (r+1)a'_{2i-1} + \sum_{i=1}^{d'/2} ra'_{2i} & (r : \text{even}) \\ -\{((r+1)/2)a'_0 + \sum_{i=1}^{d'/2} ra'_{2i-1} + \sum_{i=1}^{d'/2} (r+1)a'_{2i}\} & (r : \text{odd}) \end{cases}$$

(note that $d' = \deg \Delta_{T_{p,q}}(t^2) = 2 \deg \Delta_{T_{p,q}}(t)$). Furthermore, we note that

$$a'_i = \begin{cases} a_{i/2}(\Delta_{T_{p,q}}(t)) & (i : \text{even}) \\ 0 & (i : \text{odd}) \end{cases}.$$

Thus we have

$$\begin{aligned}
t'_0 &= \begin{cases} (r/2)\{a_0(\Delta_{T_{p,q}}(t)) + \sum_{i=1}^{d'/2} 2a_i(\Delta_{T_{p,q}}(t))\} & (r : \text{even}) \\ -(r+1)/2\{a_0(\Delta_{T_{p,q}}(t)) + \sum_{i=1}^{d'/2} 2a_i(\Delta_{T_{p,q}}(t))\} & (r : \text{odd}) \end{cases} \\
&= (-1)^r \left\lceil \frac{r}{2} \right\rceil \\
&= \frac{-1}{2} \cdot (-1)^r d_1(T_{2,2r+1}).
\end{aligned}$$

This implies that for $r > \deg \Delta_{T_{p,q}}(t^2)$, we have

$$-2t_0(\Delta_{(T_{p,q})_{2,2r+1}}(t)) = d_1(T_{2,2r+1}).$$

In particular, $(2pq \pm 1 - 1)/2 > \deg \Delta_{T_{p,q}}(t^2)$, and hence we have

$$d_1((T_{p,q})_{2,2pq \pm 1}) = -2t_0(\Delta_{(T_{p,q})_{2,2pq \pm 1}}(t)) = d_1(T_{2,2pq \pm 1}).$$

□

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